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Dirac equations in $n + 1$ dimensions

Yu Jiang[1,2]

[1] Departamento de Física, Universidad Autónoma Metropolitana-Iztapalapa, Apartado Postal 55-534, 09340 México DF, Mexico
[2] Programa de Ingeniería Molecular, Instituto Mexicano del Petróleo, Lázaro Cárdenas 152 07730 México, DF, Mexico

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Abstract
The Dirac equation in $n + 1$ dimensions is derived by a simple algebraic approach. The similarity in the structure of the arbitrary n-dimensional Dirac equations in a central field and their solutions is discussed.

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The generalization of the Schrödinger equation from three dimensions to arbitrary n dimensions is achieved by simply using the eigenvalues of the generalized orbital angular momentum L^2 in place of the three-dimensional ones [1–6]. For the Dirac equation, one has to deal with the generalized orbital and spin angular momentum operators which form the elements of a Lie group and a Lie algebra. In a recent work [7], the n-dimensional orbital angular momentum problem has been studied by the method of the group theory. Unfortunately, the main results concerning the eigenvalues of the orbital–spin angular momentum interaction operator seem to be incomplete, and do not reduce to the well-known three-dimensional results. The Dirac equation in arbitrary n dimensions was derived by using the self-adjoint ladder operator method some decades ago [8], though the radial Dirac equation and its solutions were not discussed. The purpose of this work is to present a simple algebraic derivation of the Dirac equation in n dimensions and point out the similar structure of the Dirac equations in arbitrary spatial dimensions in a central field, and show that the exact solutions of 3D Dirac equations can be generalized to n-dimensional Dirac equations in a straightforward way.

The Dirac equation for a central field in $n + 1$ dimensions can be written as

$$\mathrm{i}\hbar\frac{\partial\Psi}{\partial t} = H\Psi, \qquad H = c\sum_{j=1}^{n}\alpha_j p_j + \beta mc^2 + V(r), \tag{1}$$

where m is the mass of the particle, $V(r)$ denotes spherically symmetric central potential and n matrices α_i satisfy the anti-commutative relations

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}, \tag{2}$$

with δ_{ij} being the Kronecker delta.

The Dirac equation for a central field can be separated without approximation in spherical coordinates. This procedure involves finding the eigenfunction of the interaction of orbital and spin angular momenta. We start by defining radial momentum and velocity operators

$$p_r = r^{-1}\left(\sum_{j=1}^{n} x_j p_j - \mathrm{i}\hbar\frac{n-1}{2}\right), \tag{3}$$

$$\alpha_r = r^{-1}\left(\sum_{j=1}^{n} \alpha_j x_j\right). \tag{4}$$

Introducing the operator κ that is related to the total angular momentum,

$$\hbar\kappa_n = \beta\left(\sum_{i<j}^{n} \sigma_{ij} L_{ij} + \hbar\frac{n-1}{2}\right) \tag{5}$$

the Dirac Hamiltonian can be rewritten in the following form:

$$H = c\alpha_r p_r + \frac{\mathrm{i}\hbar c}{r}\alpha_r\beta\kappa + \beta mc^2 + V(r). \tag{6}$$

Thus, the problem of finding the eigenvalues of H is converted into finding those of κ_n. Note that the orbital angular momentum operators defined by

$$L_{ij} = x_i p_j - x_j p_i = -\mathrm{i}\left[x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i}\right], \tag{7}$$

satisfy the following algebraic relations:

$$L_{ij} = -L_{ji}, \qquad L_{ij} = L_{ij}^{\dagger}, \tag{8}$$

$$[L_{ij}, L_{ik}] = \mathrm{i}L_{jk}, \tag{9}$$

$$[L_{ij}, L_{kl}] = 0, \qquad \text{for} \quad \mathrm{i} \neq j \neq k \neq l, \tag{10}$$

$$L_{ij}L_{kl} + L_{ki}L_{jl} + L_{jk}L_{il} = 0, \qquad \text{for} \quad i \neq j \neq k \neq l, \tag{11}$$

where the indices i, j, k, l, take the values $1, 2, \ldots, n$; n being the dimension of the space. The quantities L_{ij} form the elements of a Lie algebra, which has a single Casimir invariant, namely the total orbital angular momentum

$$L^2 = \sum_{i<j}^{n} L_{ij}^2. \tag{12}$$

The generalized spin angular momentum σ_{ij} is defined by

$$\sigma_{ij} = \frac{-\mathrm{i}}{2}[\alpha_i, \alpha_j], \tag{13}$$

which satisfy the following relations:

$$\sigma_{ij} = -\sigma_{ji}, \qquad \sigma_{ij} = \sigma_{ij}^{\dagger}, \tag{14}$$

$$\sigma_{ij}^2 = 1, \tag{15}$$

$$[\sigma_{ij}, \sigma_{ik}] = i\sigma_{jk}, \qquad \text{for} \quad i \neq j \neq k, \tag{16}$$

$$[\sigma_{ij}, \sigma_{kl}] = 0, \qquad \text{for} \quad i \neq j \neq k \neq l. \tag{17}$$

We now calculate the eigenvalue of κ_n via a new operator

$$\mathcal{L} = \sum_{i<j}^{n} \sigma_{ij} L_{ij}, \tag{18}$$

so that $\kappa_n = \beta[\mathcal{L} + (n-1)/2]$. Since $[H, \mathcal{L}] = 0$, $[L^2, \mathcal{L}] = 0$, H, L^2, and $\mathcal{L}$ have common eigenfunctions, the eigenvalues of $\mathcal{L}$ can be found by establishing a relation between $\mathcal{L}$ and L^2. We start with the following quantity:

$$\mathcal{L}^2 = \sum_{i<j}^{n} \sum_{k<l}^{n} \sigma_{ij} L_{ij} \sigma_{kl} L_{kl}, \tag{19}$$

which can be calculated by dividing it into three parts. The first part is given by

$$\mathcal{L}_1^2 = \sum_{i<j}^{n} (\sigma_{ij} L_{ij})^2 = L^2, \tag{20}$$

which is a partial sum of those terms with paired equal indices, i.e., $i = k$, and $j = l$. The second sum includes all those terms that can be contracted into $\sigma_{ij} L_{ij}$ in accordance with conditions (8) and (14). They are

$$\mathcal{L}_2^2 = \sum_{i<j}^{n} \left(\sum_{k=1}^{i-1} \sigma_{ki} L_{ki} \sigma_{kj} L_{kj} + \sum_{k=j+1}^{n} \sigma_{ik} L_{ik} \sigma_{jk} L_{jk} \sum_{k=i+1}^{j-1} \sigma_{kj} L_{kj} \sigma_{ik} L_{ik} \right) = -(n-2)\mathcal{L}. \tag{21}$$

The third part accounts for all terms with unequal indices, which can be written as

$$\mathcal{L}_3^2 = \sum_{i<j}^{n} \sum_{k<l}^{n} \sigma_{ij} L_{ij} \sigma_{kl} L_{kl}, \qquad i \neq j \neq k \neq l. \tag{22}$$

Due to the symmetry of the terms in the sum (18), one finds that all terms appeared in (20) can be covered by summing up only the following triple terms:

$$\begin{aligned} &\sigma_{ij} L_{ij} \sigma_{kl} L_{kl} + \sigma_{ik} L_{ik} \sigma_{jl} L_{jl} + \sigma_{jk} L_{jk} \sigma_{il} L_{il} \\ &\qquad = \sigma_{ij} \sigma_{kl} (L_{ij} L_{kl} + L_{ki} L_{jl} + L_{jk} L_{il}) = 0, \qquad i < j < k < l. \end{aligned} \tag{23}$$

Here we have made use of (9) and (15). It can be shown that the number of terms involved in each partial sum are $N_1 = n(n-1)/2$ in $\mathcal{L}_1^2$, and $N_2 = n(n-1)(n-2)$ in $\mathcal{L}_2^2$. The number of terms in $\mathcal{L}_3^2$ is given by

$$N_3 = 6 \sum_{i<j<k<l} 1 = \frac{1}{4} n(n-1)(n-2)(n-3). \tag{24}$$

It is easy to see that $N_1 + N_2 + N_3 = N = [n(n-1)/2]^2$. From equations (20), (21) and (23), we find

$$L^2 = \mathcal{L}(\mathcal{L} + n - 2). \tag{25}$$

Since the eigenfunctions of L^2 are doubly degenerate we may write

$$\mathcal{L}\Psi_1 = l\Psi_1, \tag{26}$$

$$\mathcal{L}\Psi_2 = -(l + n - 2)\Psi_2, \tag{27}$$

which lead to

$$L^2 \Psi_i = l(l + n - 2)\Psi_i, \qquad i = 1, 2. \tag{28}$$

Thus, the eigenvalues of κ_n can be written as

$$\kappa_n = \pm\left(j + \frac{n-2}{2}\right), \qquad j = l \pm 1/2 \tag{29}$$

or

$$\kappa_n = \begin{cases} -\left(l + \frac{n-1}{2}\right), & j = l + 1/2 \\ l + \frac{n-3}{2}, & j = l - 1/2 \end{cases}$$

which are the same as derived in [9]. By introducing two-component wavefunction

$$\Psi = r^{-\frac{n-1}{2}} \begin{pmatrix} \mathrm{i}G \\ -F \end{pmatrix}$$

we obtain the radial Dirac equation in n dimensions

$$\begin{aligned} &\hbar c\frac{\mathrm{d}G}{\mathrm{d}r} + \frac{\hbar c\kappa_n}{r}G - [E + mc^2 - V(r)]F = 0, \\ &-\hbar c\frac{\mathrm{d}F}{\mathrm{d}r} + \frac{\hbar c\kappa_n}{r}F - [E - mc^2 - V(r)]G = 0, \end{aligned} \tag{30}$$

which can be reduced to the Dirac equation in two dimensions with $\kappa_2 = \pm j$, and in three dimensions with $\kappa_3 = \pm(j + 1/2)$, for $j = l \pm 1/2$.

From the structure of the radial Dirac equation in n dimensions, it follows that all the exact solutions to the 3D Dirac equations can be translated into the exact solutions of the Dirac equations in arbitrary dimensions with $n > 1$, by a simple substitution of κ by κ_n. As an example, let us analyse the solution of the D-dimensional radial equations for a Dirac particle in a Coulomb potential $V(r) = -Z\alpha/r$. (Henceforth, we use D to denote the spatial dimension in place of n, to avoid confusion with the principal quantum number.) From section 9.6 of [9], we find that the normalized radial wavefunctions are given by

$$\left.\begin{matrix} G(r) \\ F(r) \end{matrix}\right\} = \frac{\pm(2\lambda)^{3/2}}{\Gamma(2\gamma_D + 1)} \times \sqrt{\frac{(mc^2 \pm E)\Gamma(2\gamma_D + n' + 1)}{4mc^2\frac{(n'+\gamma_D)mc^2}{E}\left[\frac{(n'+\gamma_D)mc^2}{E} - \kappa_D\right]n'!}} \times (2\lambda r)^{\gamma_D}\,\mathrm{e}^{-\lambda r}\left\{\left[\frac{(n' + \gamma_D)mc^2}{E} - \kappa_D\right] \times F(-n', 2\gamma_D + 1, 2\lambda r) \mp n'F(1 - n', 2\gamma_D + 1, 2\lambda r)\right\} \tag{31}$$

with the eigenvalues

$$E = mc^2\left\{1 + \frac{(Z\alpha)^2}{\left[n - |\kappa_D| + (D-3)/2 + \sqrt{\kappa_D^2 - (Z\alpha)^2}\right]^2}\right\}^{-1/2}, \tag{32}$$

where $n' = 0, 1, 2, \ldots$, and the principal quantum number is defined by $n = n' + |\kappa_D| - (D - 3)/2 = 1, 2, \ldots$. The other parameters are defined by

$$\lambda = \frac{(m^2c^4 - E^2)^{1/2}}{\hbar c}, \tag{33}$$

$$\gamma_D = \kappa_D^2 - (Z\alpha)^2 = \left(j + \frac{D-2}{2}\right)^2 - (Z\alpha)^2. \tag{34}$$

Another interesting case is to find the stationary continuum state of a Dirac particle in a Coulomb field, in arbitrary D dimensions. Generalizing the derivation in section 9.9 of [9] to D dimensions, we find the wavefunction

$$G = \frac{C_1(2pr)^{\gamma_D} e^{\frac{\pi\lambda}{2}} |\Gamma(\gamma_D + i\lambda)|}{2(\pi p)^{1/2}\Gamma(2\gamma_D + 1)} \{e^{-ipr+i\eta_D}(\gamma_D + i\lambda)F(\gamma_D + 1 + i\lambda, 2\gamma_D + 1, 2ipr) + c.c.\}, \tag{35}$$

$$F = \frac{C_2(2pr)^{\gamma_D} e^{\frac{\pi\lambda}{2}} |\Gamma(\gamma_D + i\lambda)|}{2(\pi p)^{1/2}\Gamma(2\gamma_D + 1)} \{e^{-ipr+i\eta_D}(\gamma_D + i\lambda)F(\gamma_D + 1 + i\lambda, 2\gamma_D + 1, 2ipr) - c.c.\}, \tag{36}$$

and the Coulomb phase shift

$$\delta_D = y \ln(2pr) - \arg\Gamma(\gamma_D + iy) - \frac{\pi\gamma_D}{2} + \eta_D, \tag{37}$$

where

$$p = \frac{(E^2 - m^2c^4)^{1/2}}{\hbar c} = i\lambda, \tag{38}$$

$$\gamma_D^2 = \kappa_D^2 - (Z\alpha)^2, \tag{39}$$

$$y = \frac{Z\alpha E}{\hbar c p}, \tag{40}$$

$$e^{2i\eta_D} = \frac{\kappa_D - iymc^2/E}{\gamma_D + iy} \tag{41}$$

and C_1 and C_2 are normalization constants.

In conclusion, we have derived the n-dimensional Dirac equation in a central field, which can be reduced to the well-known forms of $D = 2$ and $D = 3$. By the similarity in the structure of the differential equation, the formal solutions of the D-dimensional Dirac equation can be obtained directly from those of 3D Dirac equations for exactly soluble potentials, with minimal modifications in those parameters that are determined by the spatial dimension n.

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